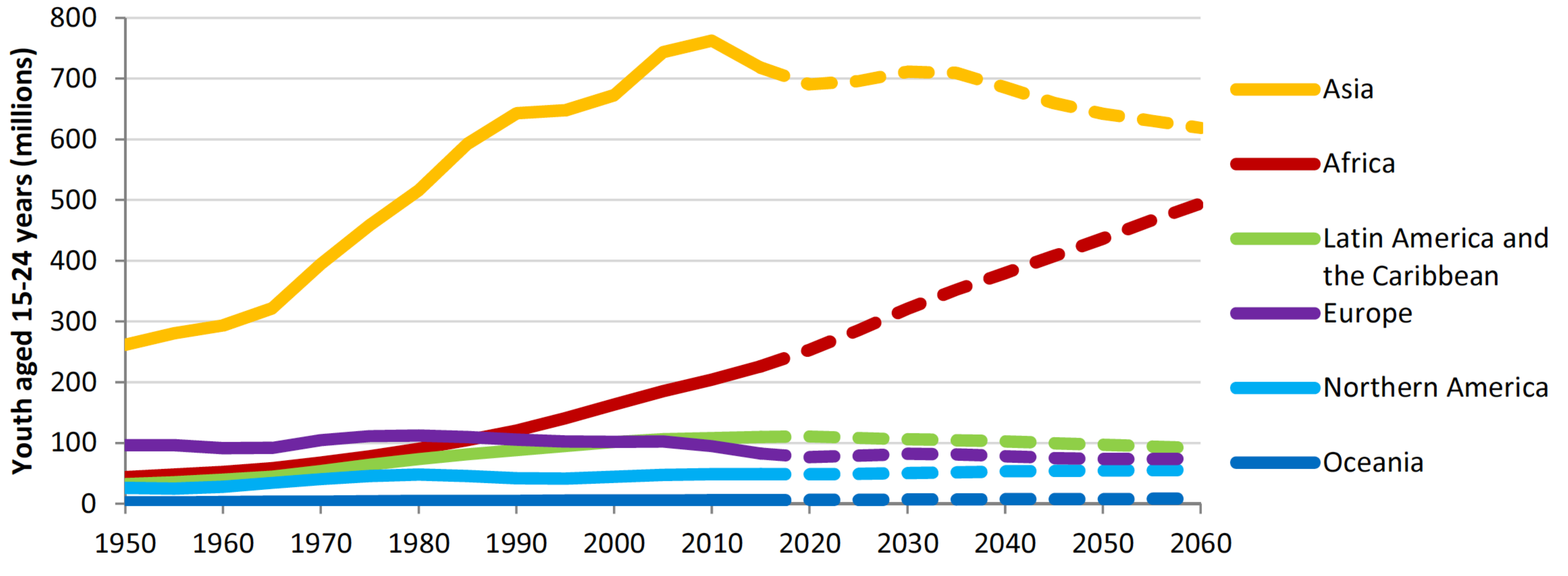
Fun With Path Integrals

Neil Turok
AIMS

collaborators: J. Feldbrugge, A. Fertig, J-L Lehners, L. Sberna, U-L. Pen

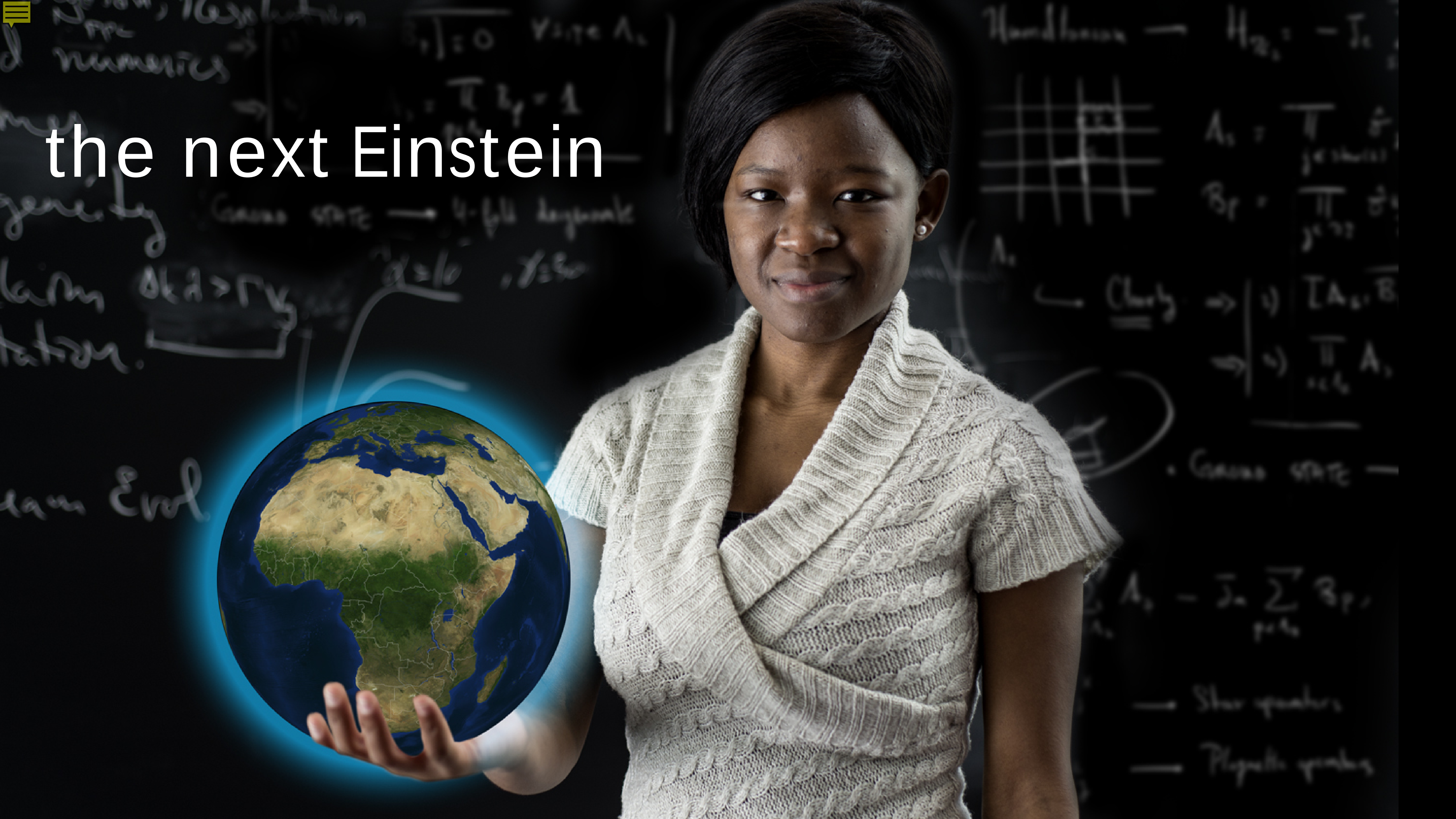
Future of Science

Figure 1: Youth aged 15-24 years, by region, 1950-2060



Data source: United Nations (2013) *World Population Prospects: The 2012 Revision.*

AIMS


the next Einstein

Scientific progress is the discovery of a more and more comprehensive simplicity

Georges Lemaître

all known physics

Schrödinger Feynman Einstein Maxwell-Yang-Mills Kobayashi-Maskawa Lagrange

$$\Psi = \int e^{\frac{i}{\hbar}\int\left(\frac{R}{16\pi G} - \frac{1}{4}F^2 + \bar{\psi} i \not{D} \psi - \lambda H \bar{\psi}\psi + |DH|^2 - V(H)\right)}$$

Euler Planck Newton Dirac Yukawa Higgs

dark energy

$$\psi = (u_L, d_L, u_R, d_R, u_L, d_L, u_R, d_R, u_L, d_L, u_R, d_R, e_L, \nu_L, e_R, \nu_R) \times 3$$

$$M\nu_R\nu_R$$

dark matter?

Highly oscillatory integrals $\Psi = \left(\frac{\nu}{\pi}\right)^{\frac{D}{2}} \int d^D\vec{x}\, e^{i\nu\Phi(\vec{x})}$

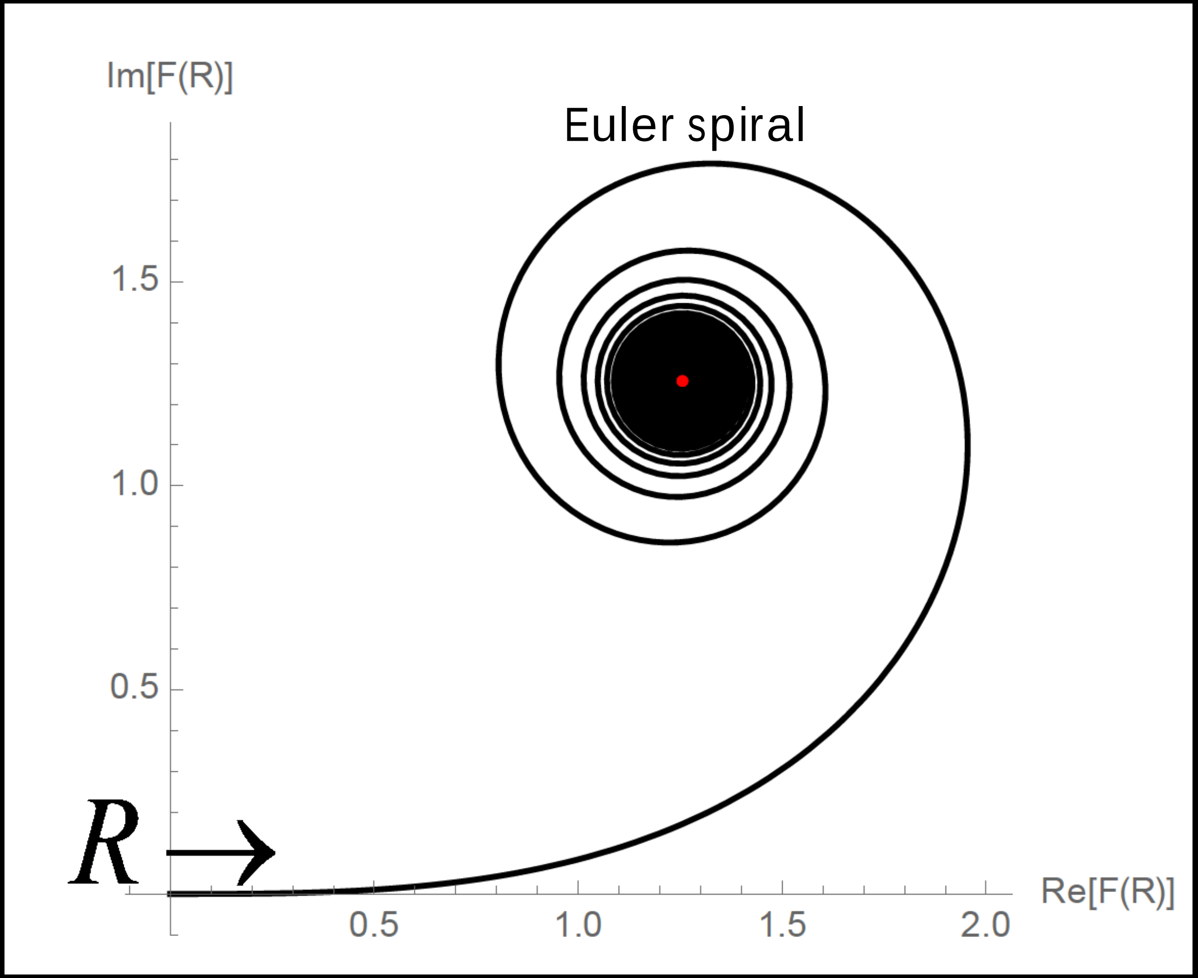
Fresnel integral

$$F(R) = \int_{-R}^{+R} e^{ix^2}\, dx$$

$$I = \lim_{R\to\infty} F(R) = e^{i\frac{\pi}{4}}\sqrt{\pi}$$

Conditionally, not absolutely convergent

Higher dimensional case?



2d: square cutoff

$$\iint dx\, dy\, e^{i(x^2+y^2)} = I^2 = i\pi$$

2d: round cutoff

$$\lim_{R\to\infty} 2\pi \int_0^R R\, dR\, e^{iR^2} = \lim_{R\to\infty} \tfrac{\pi}{i}\left(e^{iR^2} - 1\right)$$

???

Resolution: use complex analyticity and Cauchy's theorem

$\lfloor x$

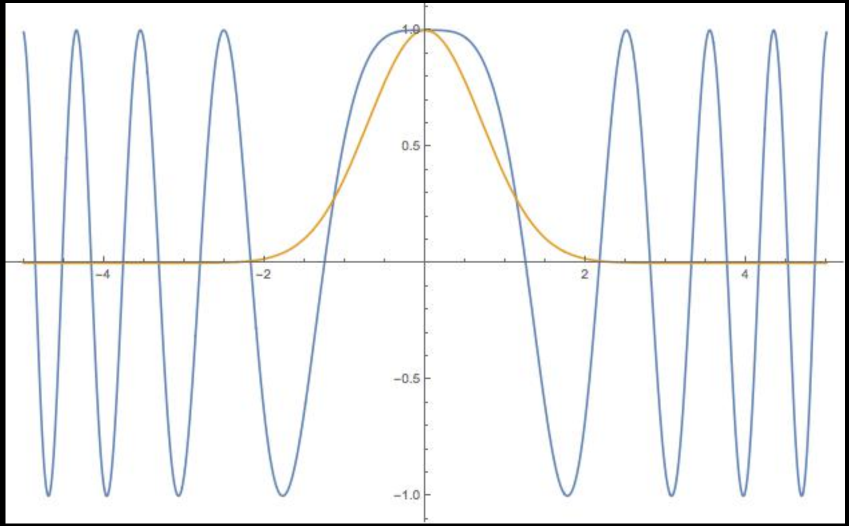
$$\left| \int R \, d\phi \, e^{iR^2 e^{2i\phi}} \right| < \frac{1}{R}$$

$$e^{i\frac{\pi}{4}} \int_{-R}^{R} ds \, e^{-s^2}$$

$-R$ $\qquad$ $+R$

Lefschetz thimble

$$x = e^{i\frac{\pi}{4}} s$$



Deforming the integration contour renders the integral *absolutely* convergent: arcs at infinity vanish as $R \to \infty$

higher dimensions:
Picard-Lefschetz

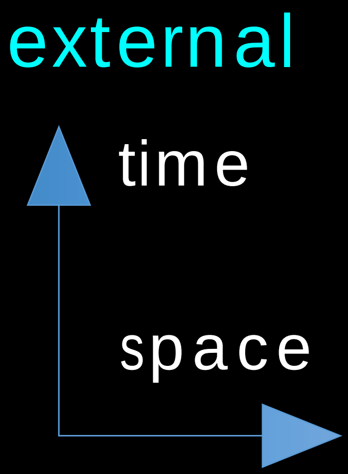
 

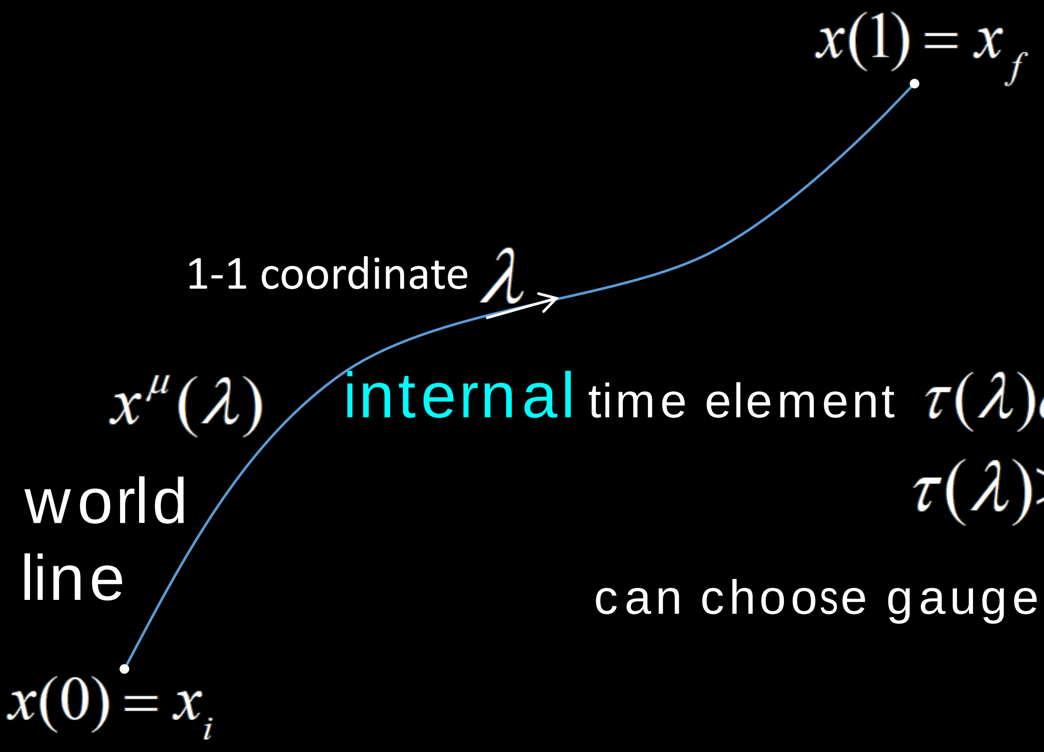
general method for doing highly oscillatory integrals:

``flow” the original contour onto a series of relevant “Lefschetz thimbles”

generalizes to arbitrary finite dimension

path integral for a relativistic particle





internal time element $\tau(\lambda)d\lambda,\ \tilde{\tau}(\tilde{\lambda})d\tilde{\lambda} = \tau(\lambda)d\lambda,\ \frac{d\tilde{\lambda}}{d\lambda} > 0$

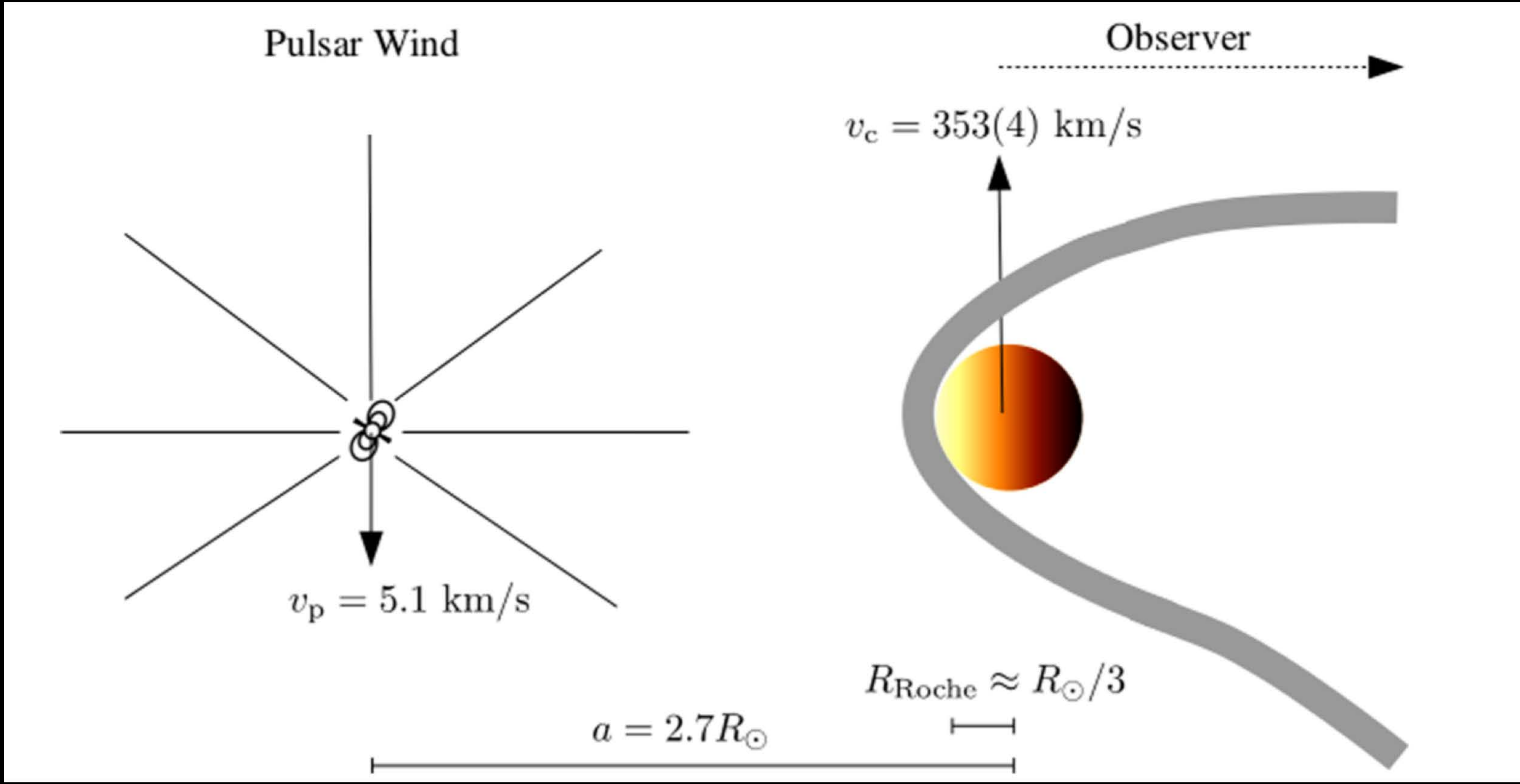
$\tau(\lambda)>0$ if nondegenerate

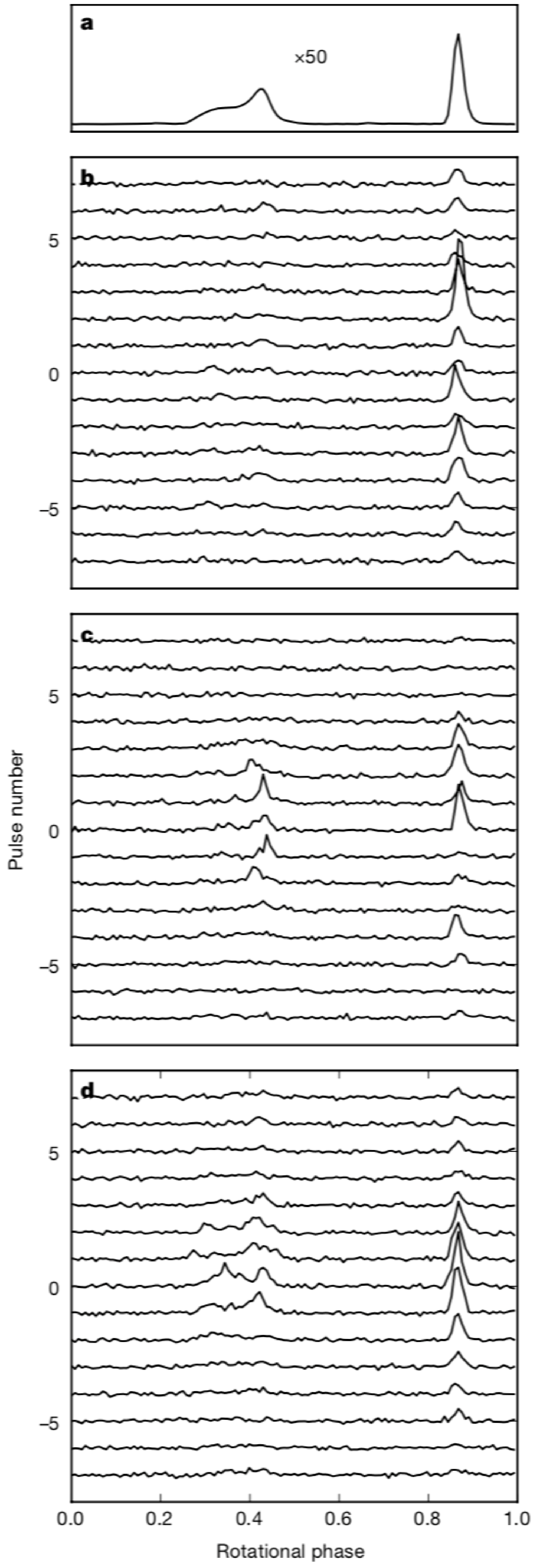
can choose gauge $\tau = const$

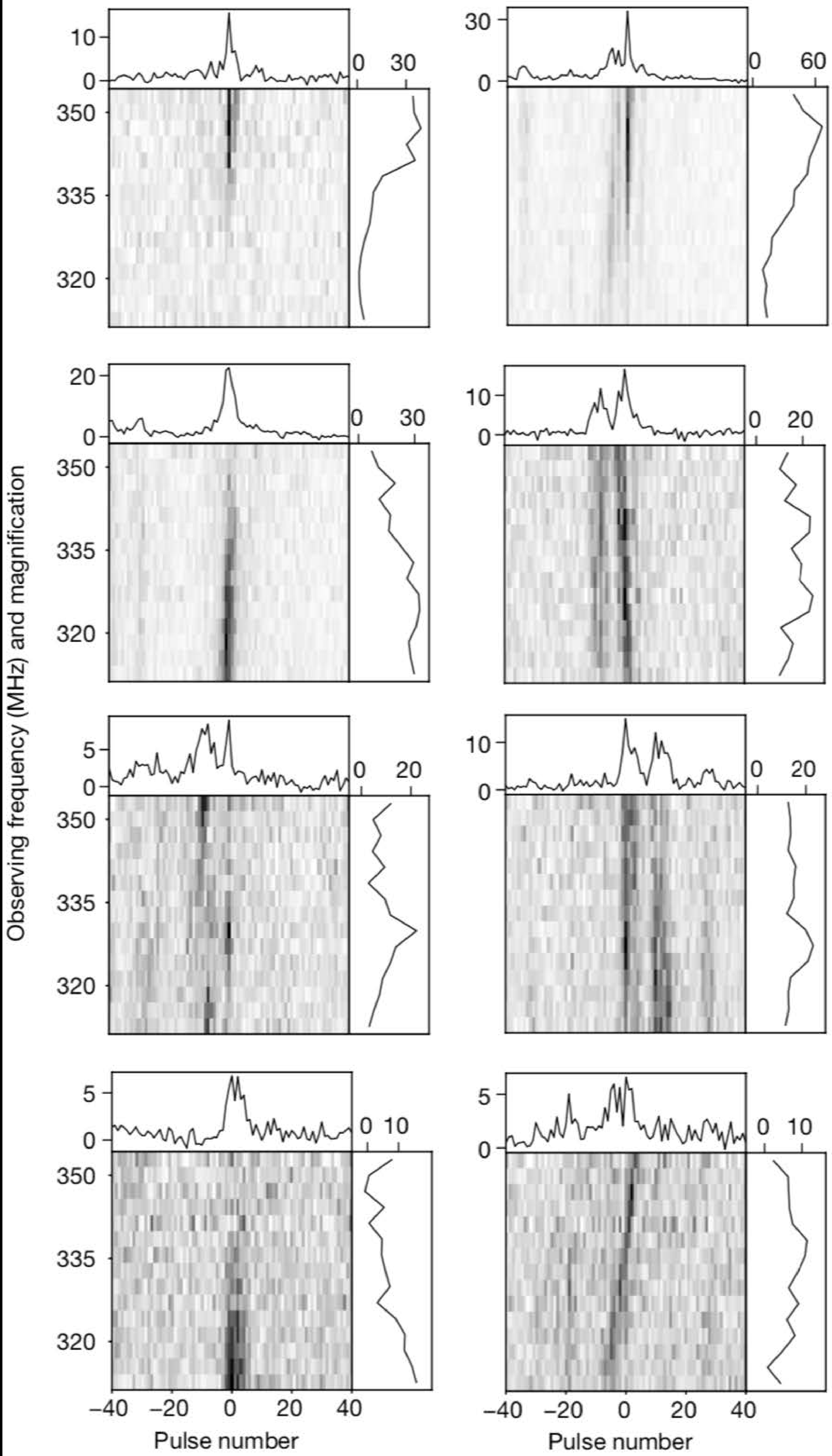
application to radio astronomy

Plasma lensing of the “black widow” pulsar PL1957+20

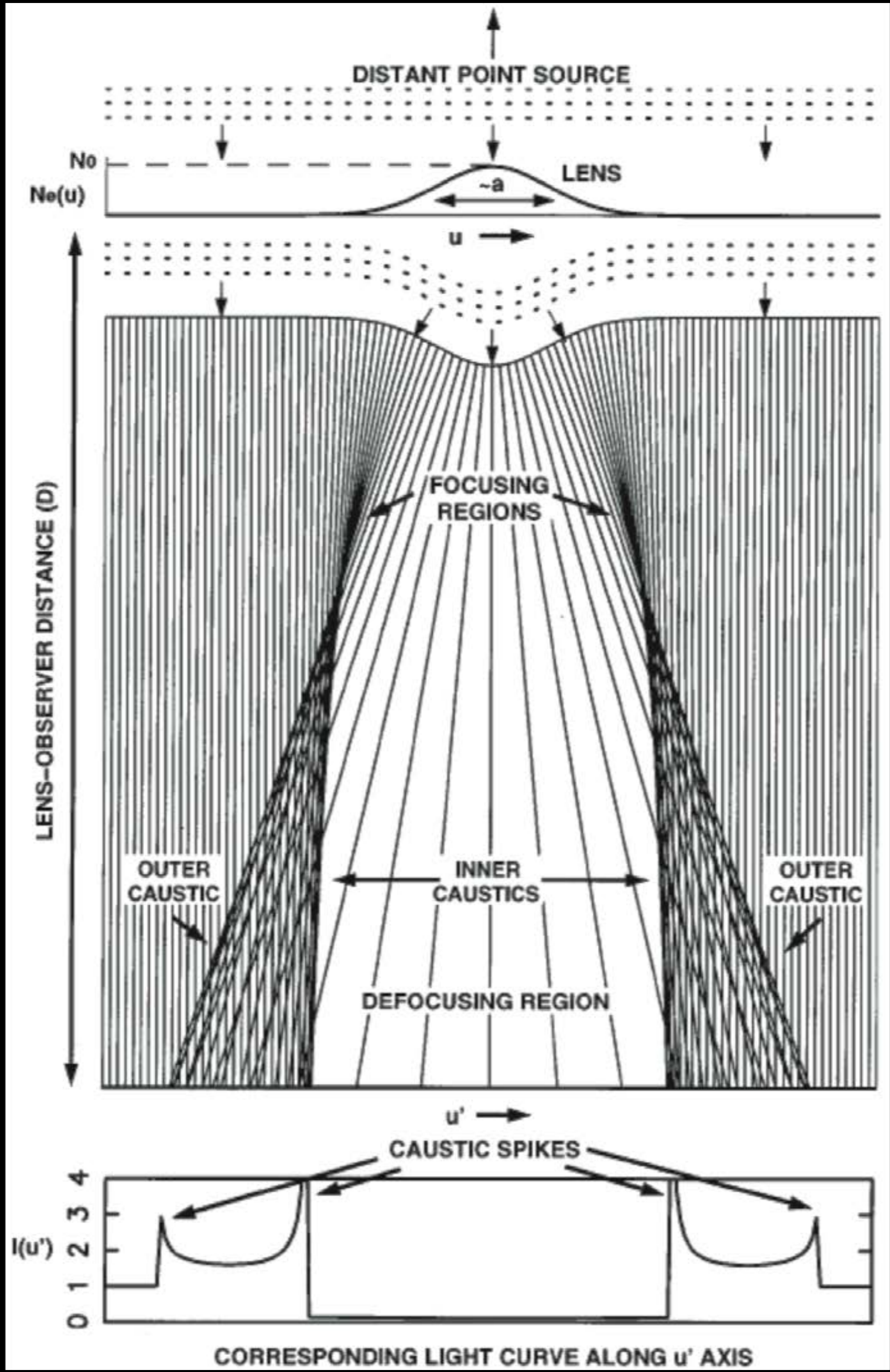
R. Main et al 2018




a
×50
b
c
d
Pulse number
5
0
−5
Rotational phase
0.0
0.2
0.4
0.6
0.8
1.0



Observing frequency (MHz) and magnification
Pulse number
Pulse number

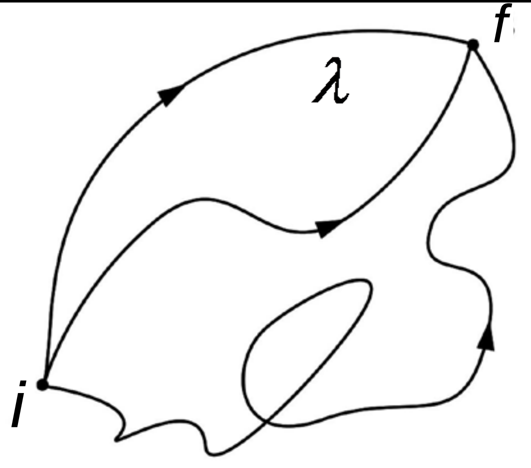


longstanding idea that there may be a big selection effect in the brightest events observed, due to lensing and magnification *e.g.*, FRBs

in plasma lensing we are always in the wave, not geometric optics regime

Feynman path integral $\Longrightarrow$ Fermat's principle of least time

quantum propagator: $$\int_{0^+}^{\infty} d\tau \int_{x_i}^{x_f} Dx\, e^{iS/\hbar}$$



action for a massless particle in a variable-c medium

$$S = \int_0^1 d\lambda \; \frac{-\left(\dot{x}^0\right)^2 + \dot{\vec{x}}^2 / c(\vec{x})^2}{4\tau}; \; \frac{\delta S}{\delta \tau} = 0 \Rightarrow \left(\frac{d\vec{x}}{dx^0}\right)^2 = c(\vec{x})^2$$

monochromatic beam: fix the initial ***energy*** and the final time.

Pass to Hamiltonian, add boundary term

$$p_0 = -\frac{\dot{x}^0}{2\tau}(= -E), \quad \vec{p} = \frac{\dot{\vec{x}}}{2\tau c^2};$$

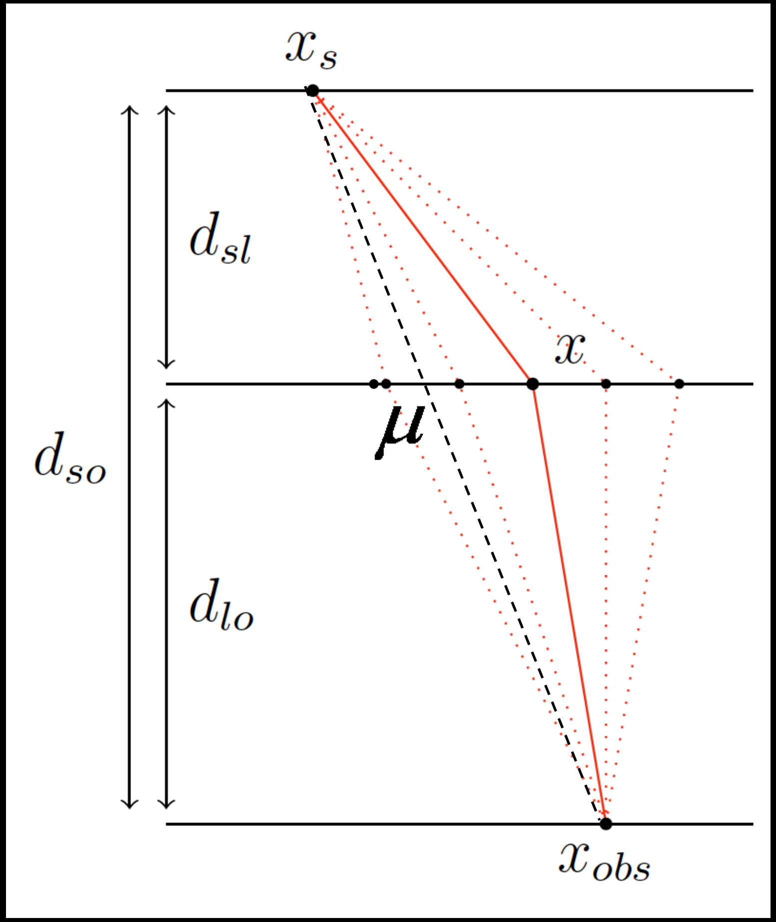
$$S = p_0 x_i^0 + \int d\lambda [p_0 \dot{x}^0 + \vec{p} \cdot \dot{\vec{x}} - \tau(-p_0^2 + \vec{p}^2)]$$

Using equations of motion, only surface term contributes. It yields a phase,

$$e^{i\frac{ET}{\hbar}} = e^{i\omega T};$$ minimizing this phase *is* Fermat's principle

$$(T \equiv x_f^0 - x_i^0 \text{ with } x_f^0 \text{ fixed})$$

thin lens



refractive index

$$\int d\vec{x}_\perp e^{i\omega \int \left| \frac{d\vec{x}}{c(\vec{x})} \right|} \qquad c(\vec{x}) = c_0 / n(\vec{x})$$

Pythagoras

Refraction

$$\approx \int d\vec{x}_\perp e^{i\frac{\omega}{2c}\left[\frac{(\vec{x}_\perp - \vec{\mu})^2}{\bar{d}} - \int dz \frac{\omega_p^2(\vec{x}_\perp, z)}{\omega^2} \right]}$$

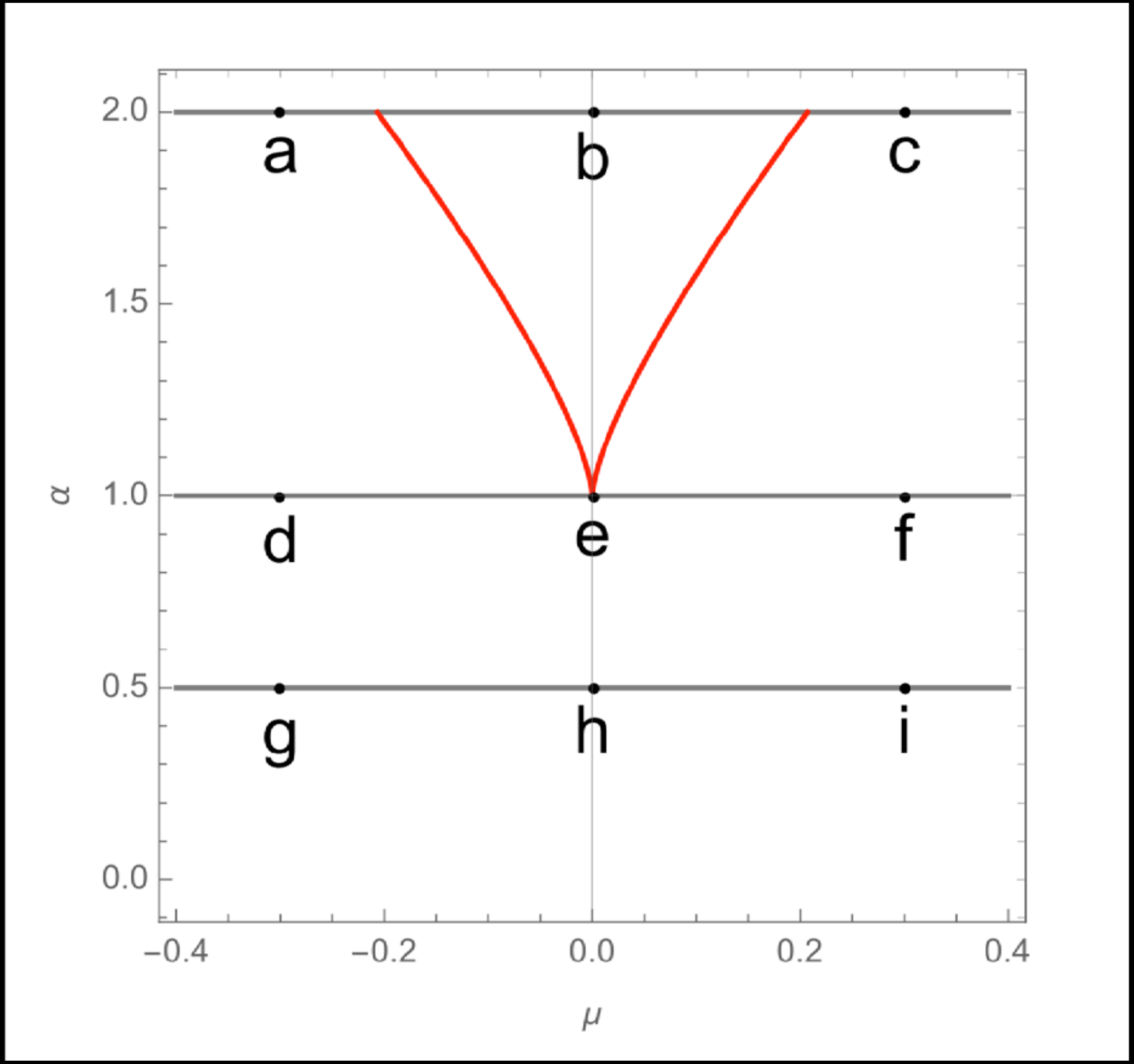
$$\frac{1}{\bar{d}} \equiv \frac{1}{d_{sl}} + \frac{1}{d_{lo}}; \quad n = \sqrt{1 - \frac{\omega_p^2}{\omega^2}} \; ; \quad \omega_p^2 = \frac{n_e(\vec{x}) e^2}{\varepsilon_0 m_e}$$

Relative amplitude (compared to no lens)

$$\Psi = \left(\frac{\nu}{\pi}\right)^{\frac{D}{2}} \int d^D\vec{x}\, e^{i\nu\underbrace{\left[\left(\vec{x}-\vec{\mu}\right)^2+\varphi(\vec{x})\right]}_{\Phi(\vec{x})}}$$

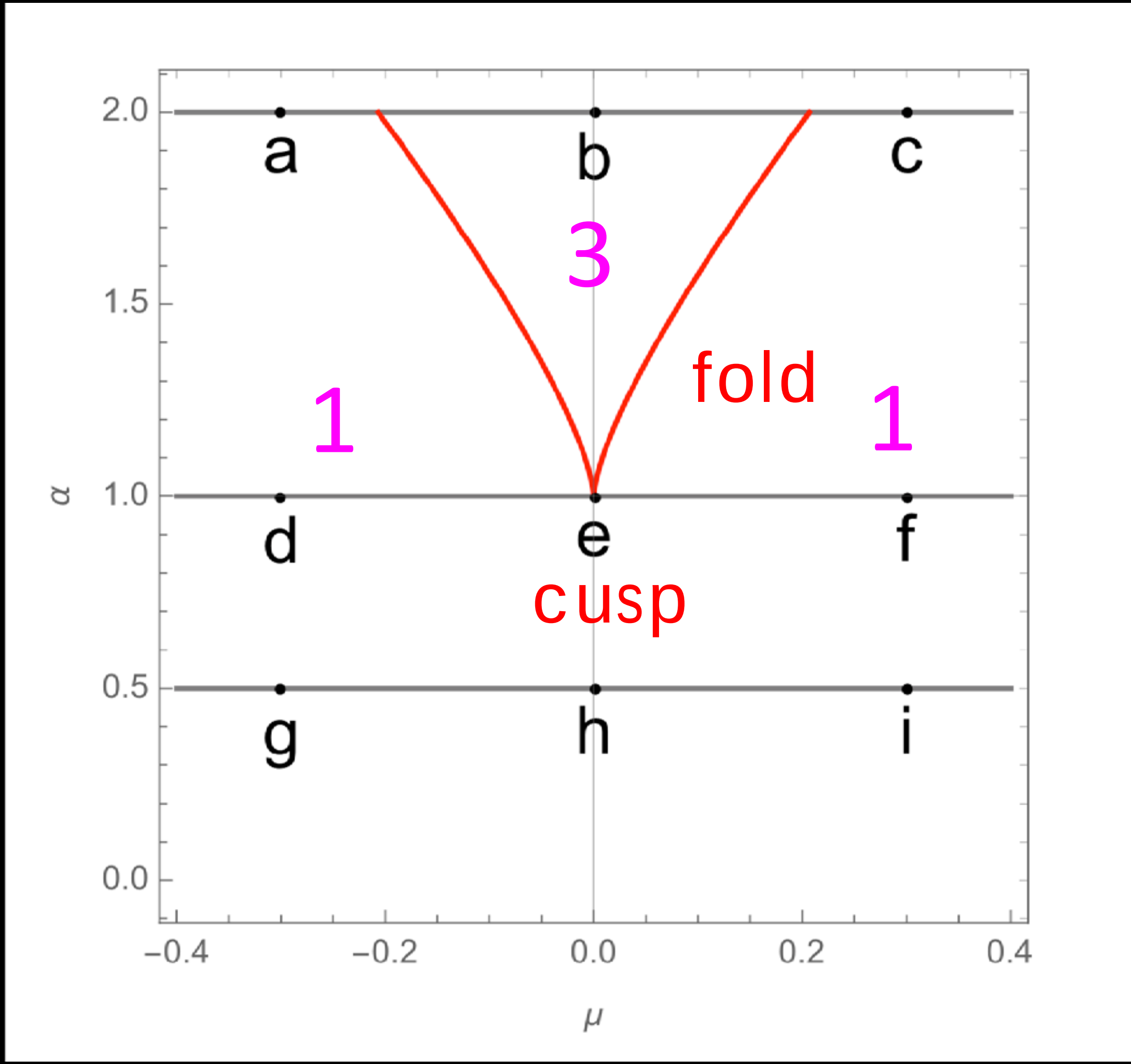
with $\vec{x}$, $\vec{\mu}$, ν expressed in dimensionless units.

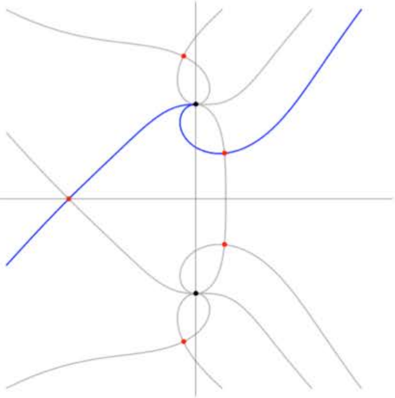
e.g. 1d localised lens: $\Phi(x) = (x-\mu)^2 + \frac{\alpha}{(1+x^2)}$



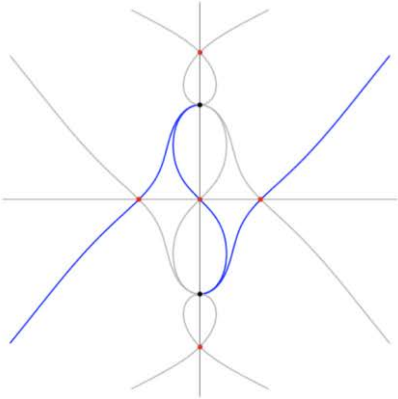
geometric optics

number of images = number of real saddles

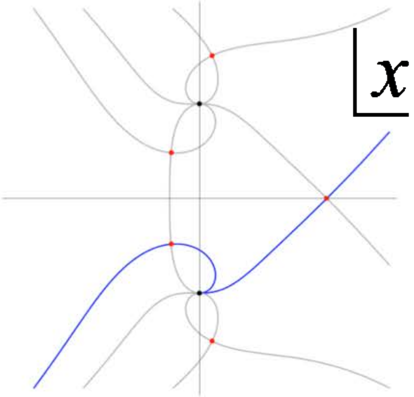




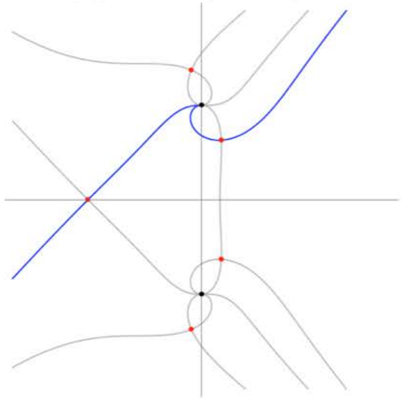
(a) $\alpha = 2, \mu < -\mu_c$



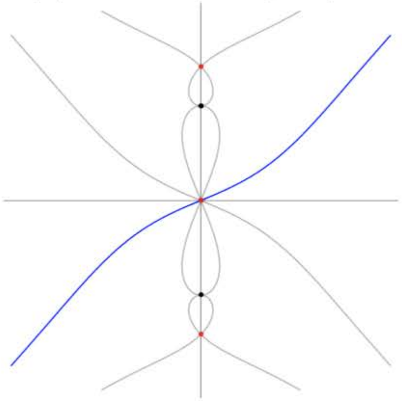
(b) $\alpha = 2, -\mu_c < \mu < \mu_c$



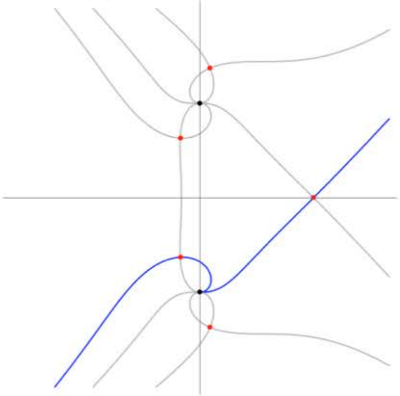
(c) $\alpha = 2, \mu > \mu_c$



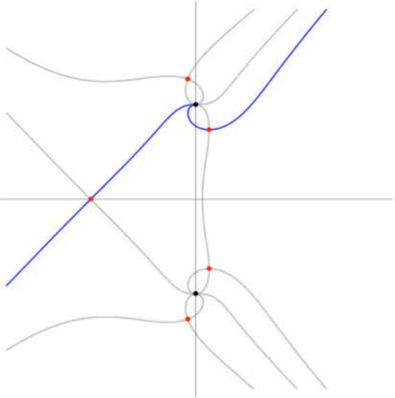
(d) $\alpha = 1, \mu < -\mu_c$



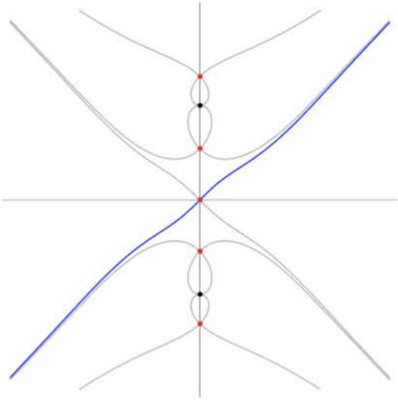
(e) $\alpha = 1, \mu = \mu_c$



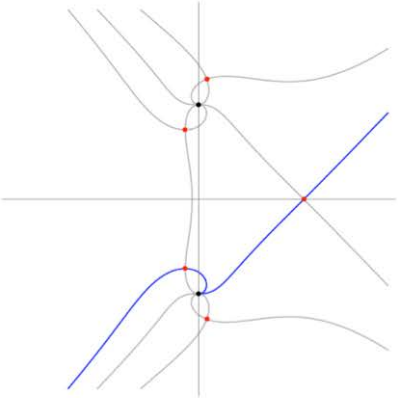
(f) $\alpha = 1, \mu > \mu_c$



(g) $\alpha = 1/2, \mu < 0$

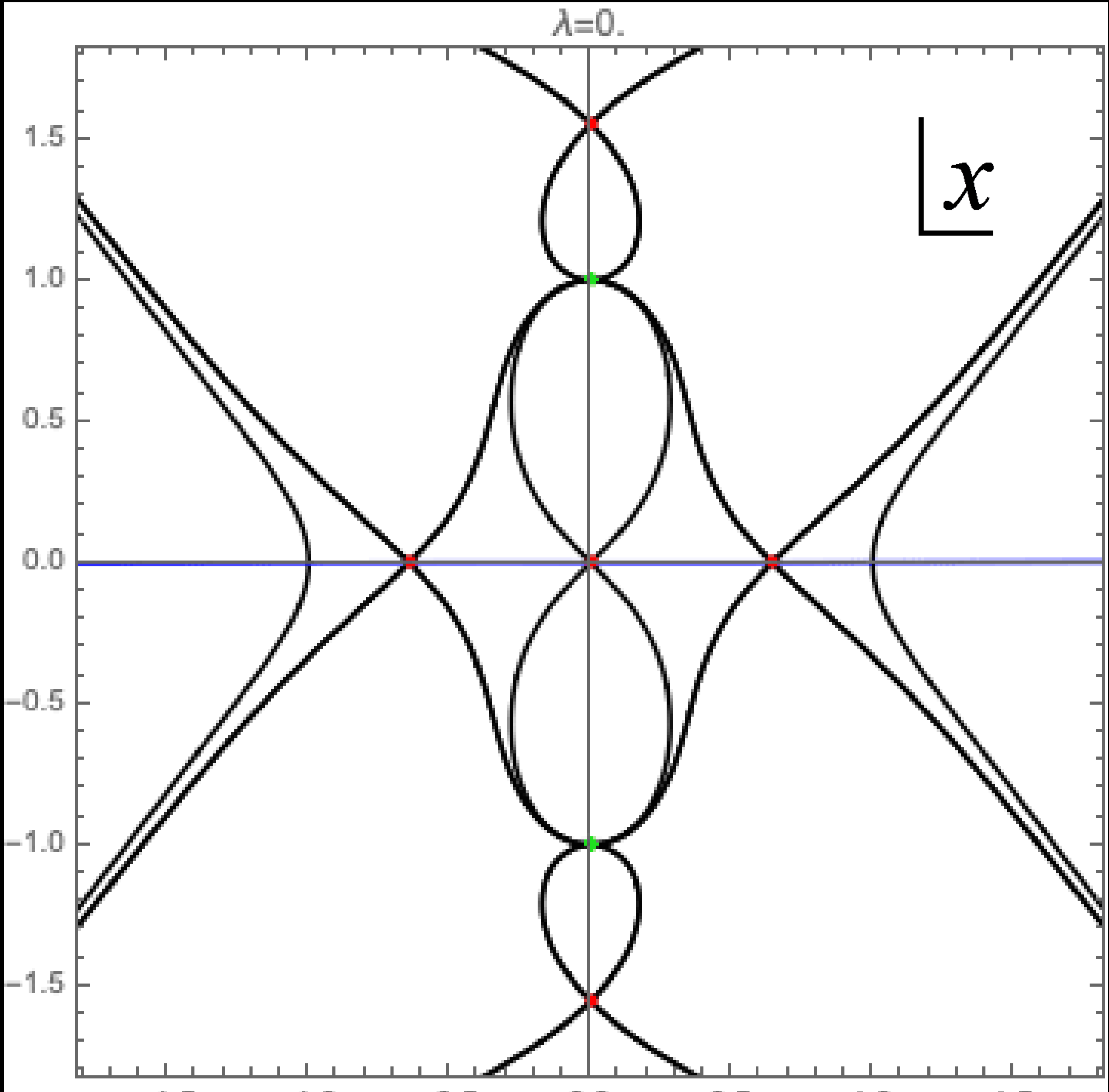


(h) $\alpha = 1/2, \mu = 0$

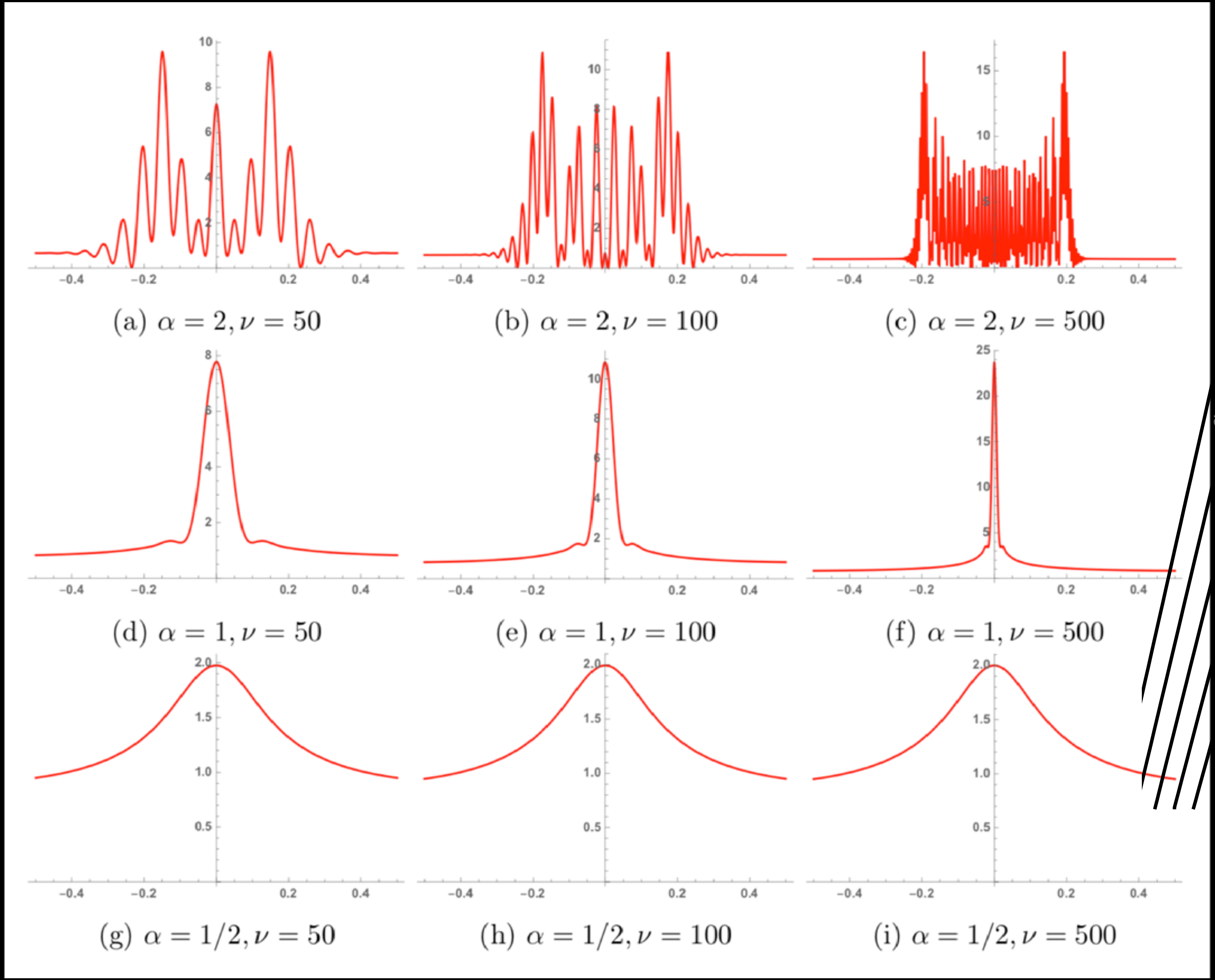


(i) $\alpha = 1/2, \mu > 0$

Flowing
the contour
(case b)

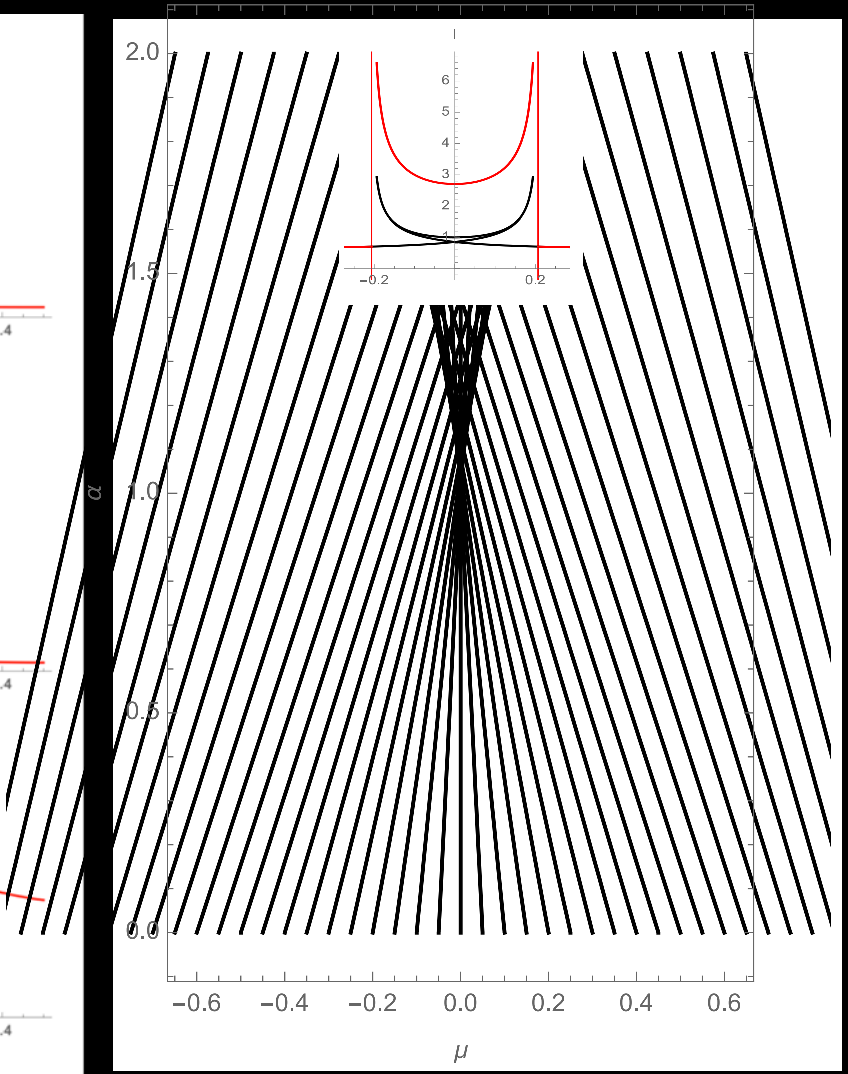


Interference patterns at increasing ν geometric optics limit


(a) $\alpha = 2, \nu = 50$ (b) $\alpha = 2, \nu = 100$ (c) $\alpha = 2, \nu = 500$

(d) $\alpha = 1, \nu = 50$ (e) $\alpha = 1, \nu = 100$ (f) $\alpha = 1, \nu = 500$

(g) $\alpha = 1/2, \nu = 50$ (h) $\alpha = 1/2, \nu = 100$ (i) $\alpha = 1/2, \nu = 500$



Things get more interesting in higher dimensions – both of the lens and of the observational parameters

2d or 3d lenses

Source sky coordinate and frequency provide two parameters, possibility of adding a second sky coordinate

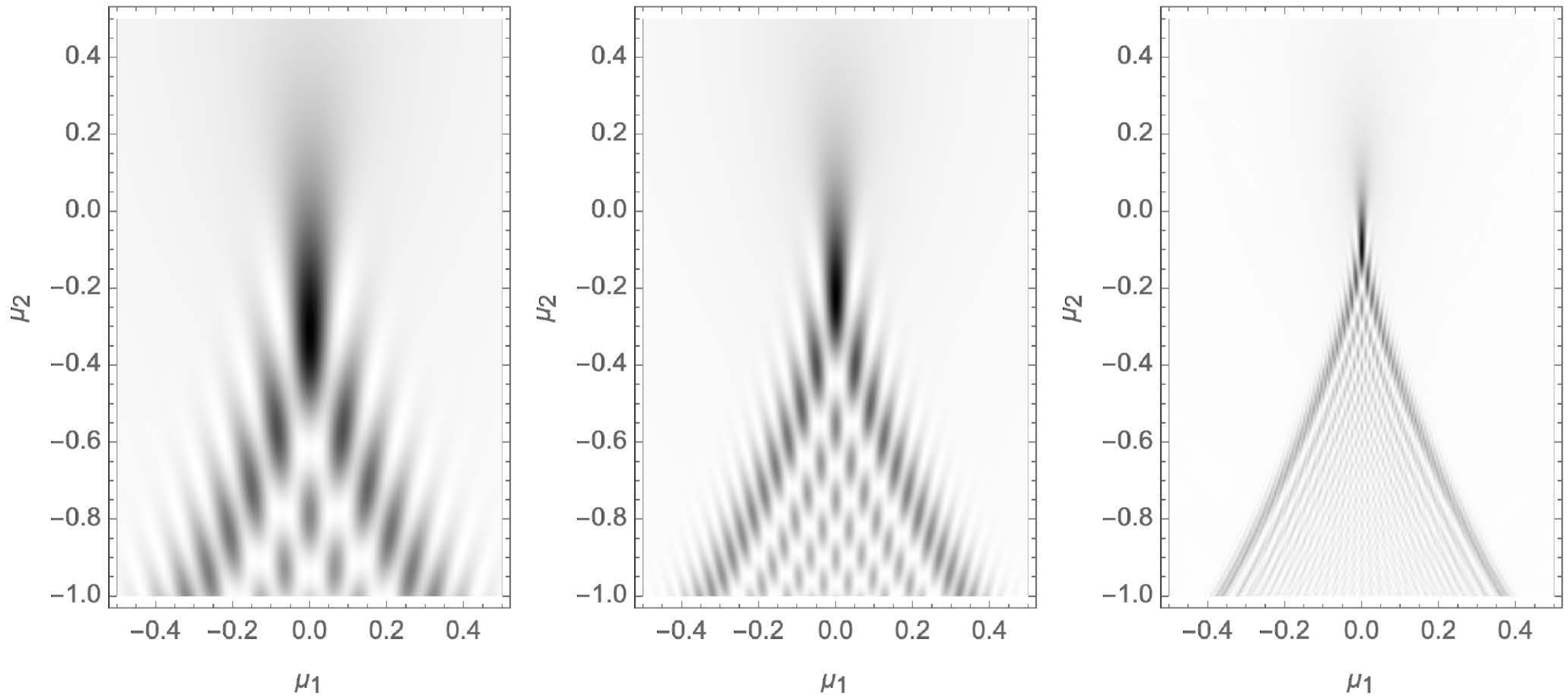
Local caustics have all been classified (invariant under diffeos)
- this is known as catastrophe theory

elementary catastrophes

Name	Symbol	K	N	$\Phi(\boldsymbol{x};\boldsymbol{\mu})$
Maximum/minimum	$A_1^{\pm}$	0	1	$\pm x^2$
Fold	A_2	1	1	$x^3/3 + \mu x$
Cusp	A_3	2	1	$x^4/4 + \mu_2 x^2/2 + \mu_1 x$
Swallowtail	A_4	3	1	$x^5/5 + \mu_3 x^3/3 + \mu_2 x^2/2 + \mu_1 x$
Elliptic umbilic	D_4^{-}	3	2	$x_1^3 - 3x_1 x_2^2 - \mu_3(x_1^2 + x_2^2) - \mu_2 x_2 - \mu_1 x_1$
Hyperbolic umbilic	D_4^{+}	3	2	$x_1^3 + x_2^3 - \mu_3 x_1 x_2 - \mu_2 x_2 - \mu_1 x_1$
Butterfly	A_5	4	1	$x^6/6 + \mu_4 x^4/4 + \mu_3 x^3/3 + \mu_2 x^2/2 + \mu_1 x$
Parabolic umbilic	D_5	4	2	$x_1^4 + x_1 x_2^2 + \mu_4 x_2^2 + \mu_3 x_1^2 + \mu_2 x_2 + \mu_1 x_1$

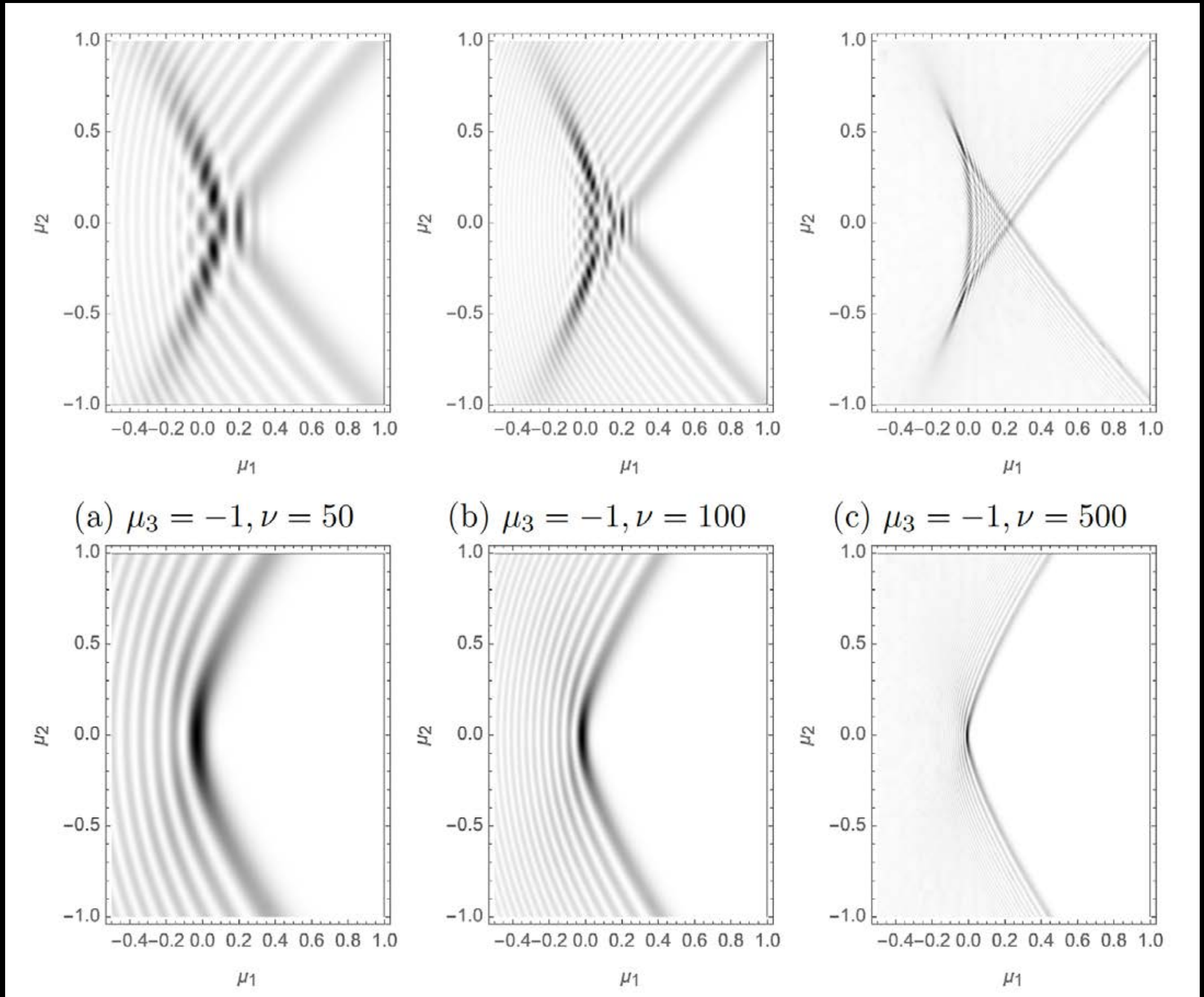
$$
\begin{tikzcd}
A_1 & A_2 \arrow[l] & A_3 \arrow[l] & A_4 \arrow[l] & A_5 \arrow[l] \\
 & & & D_4 \arrow[ul] & D_5 \arrow[l] \arrow[ul]
\end{tikzcd}
$$

cusp



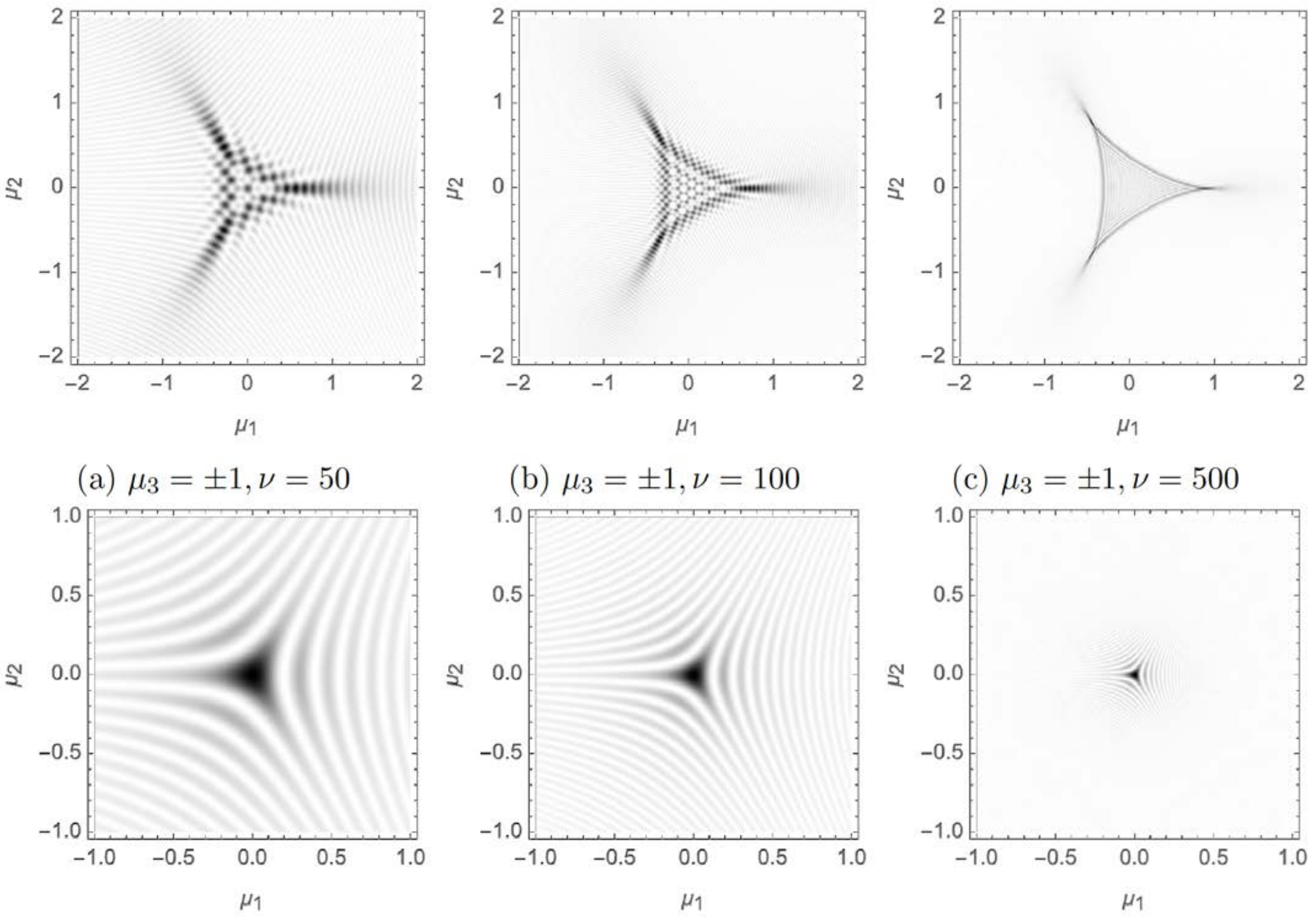
(a) $\nu = 50$ (b) $\nu = 100$ (c) $\nu = 500$

swallowtail



(a) $\mu_3 = -1, \nu = 50$ (b) $\mu_3 = -1, \nu = 100$ (c) $\mu_3 = -1, \nu = 500$

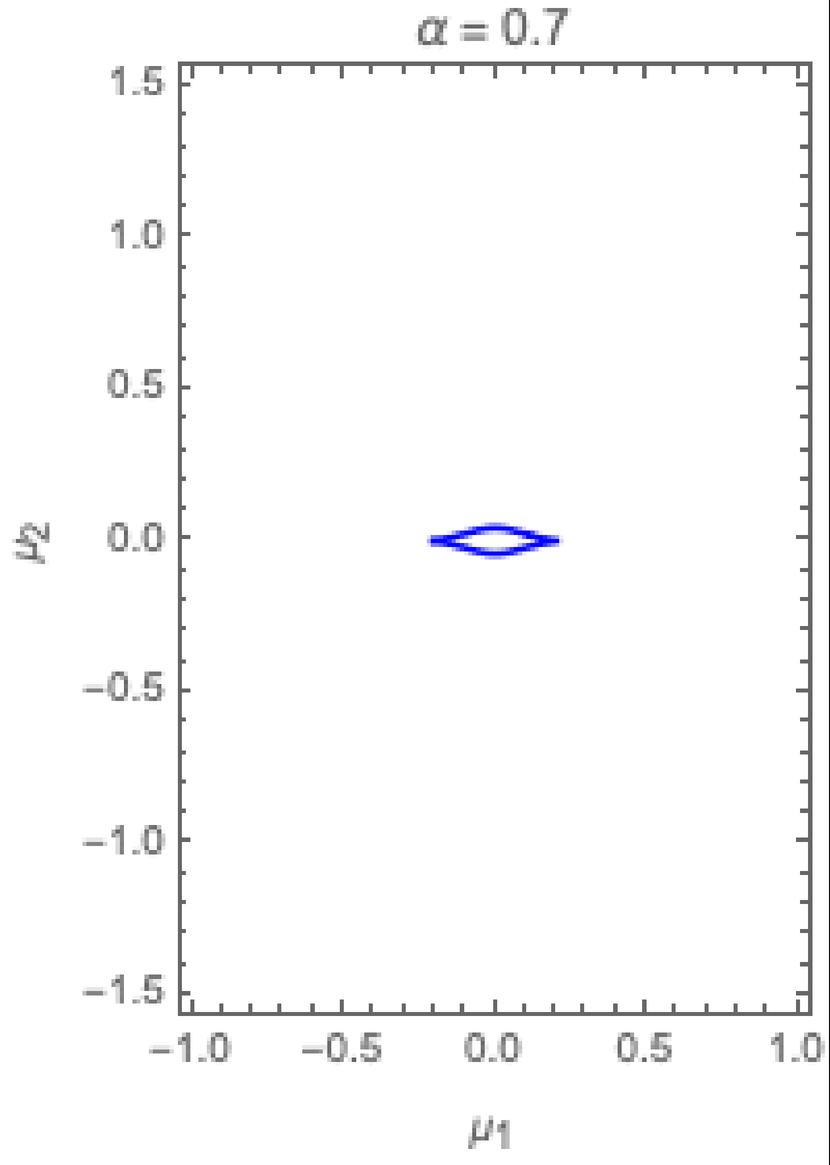
elliptic umbilic



(a) $\mu_3 = \pm 1, \nu = 50$ (b) $\mu_3 = \pm 1, \nu = 100$ (c) $\mu_3 = \pm 1, \nu = 500$

(d) $\mu_3 = 0, \nu = 50$ (e) $\mu_3 = 0, \nu = 100$ (f) $\mu_3 = 0, \nu = 500$

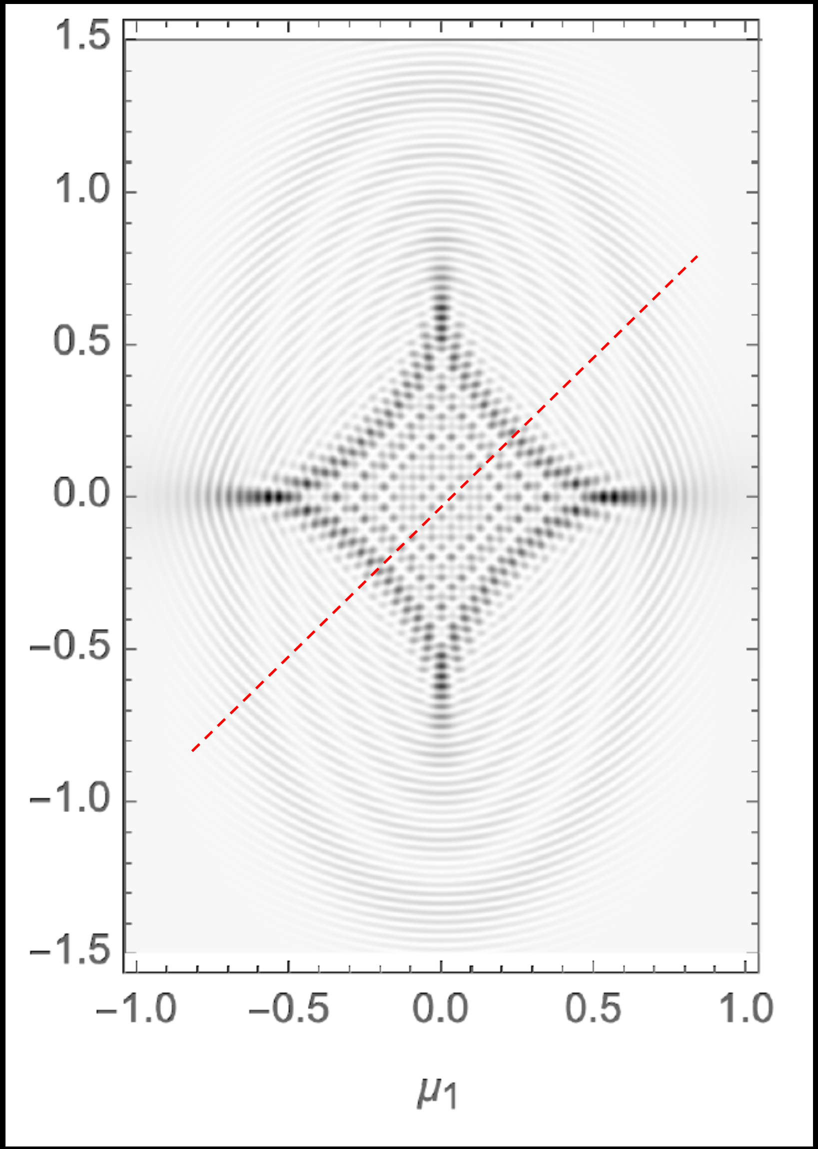
back to localised lenses

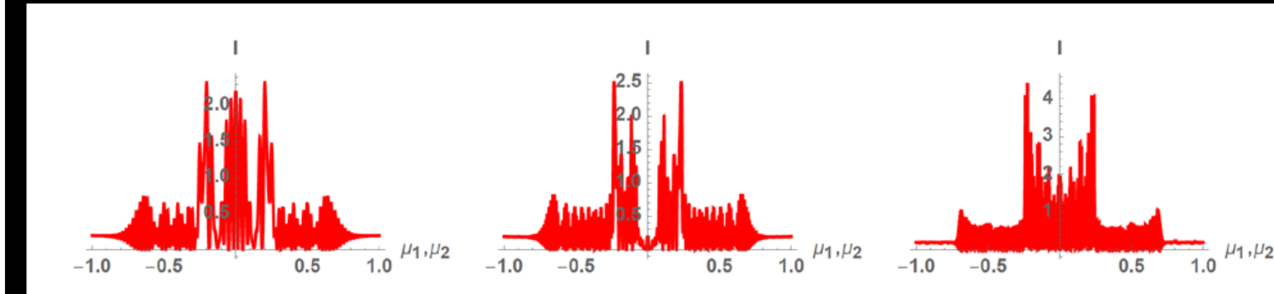


$2d:$

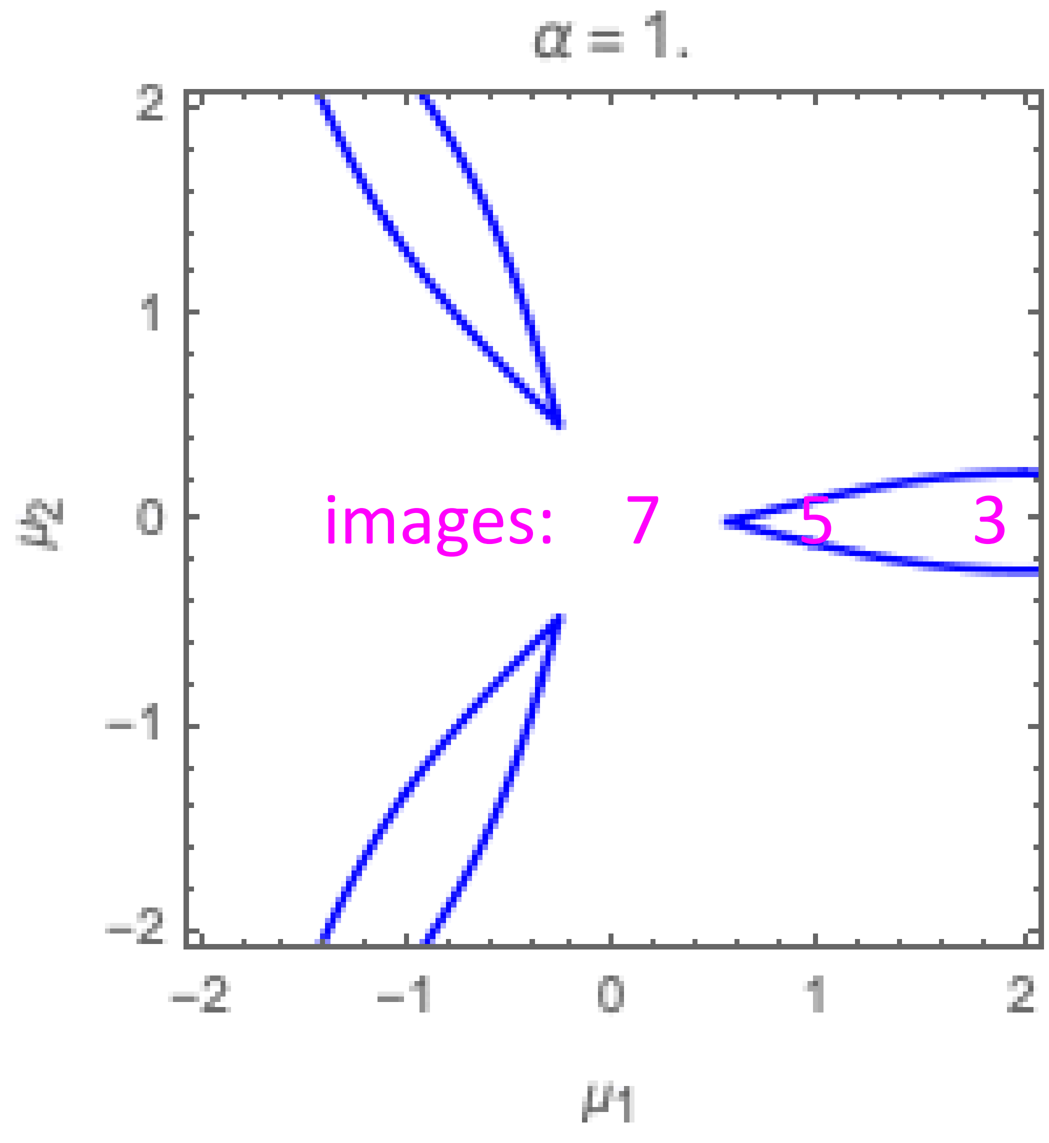
$$\Phi(\vec{x}) = (\vec{x} - \vec{\mu})^2 + \frac{\alpha}{(1+x_1^2+2x_2^2)}$$

As α increases, evolves from pair of twin cusps with folds, through hyperbolic umbilic to create an elliptical fold with no cusps





(a) $\nu = 50$ (b) $\nu = 100$ (c) $\nu = 500$

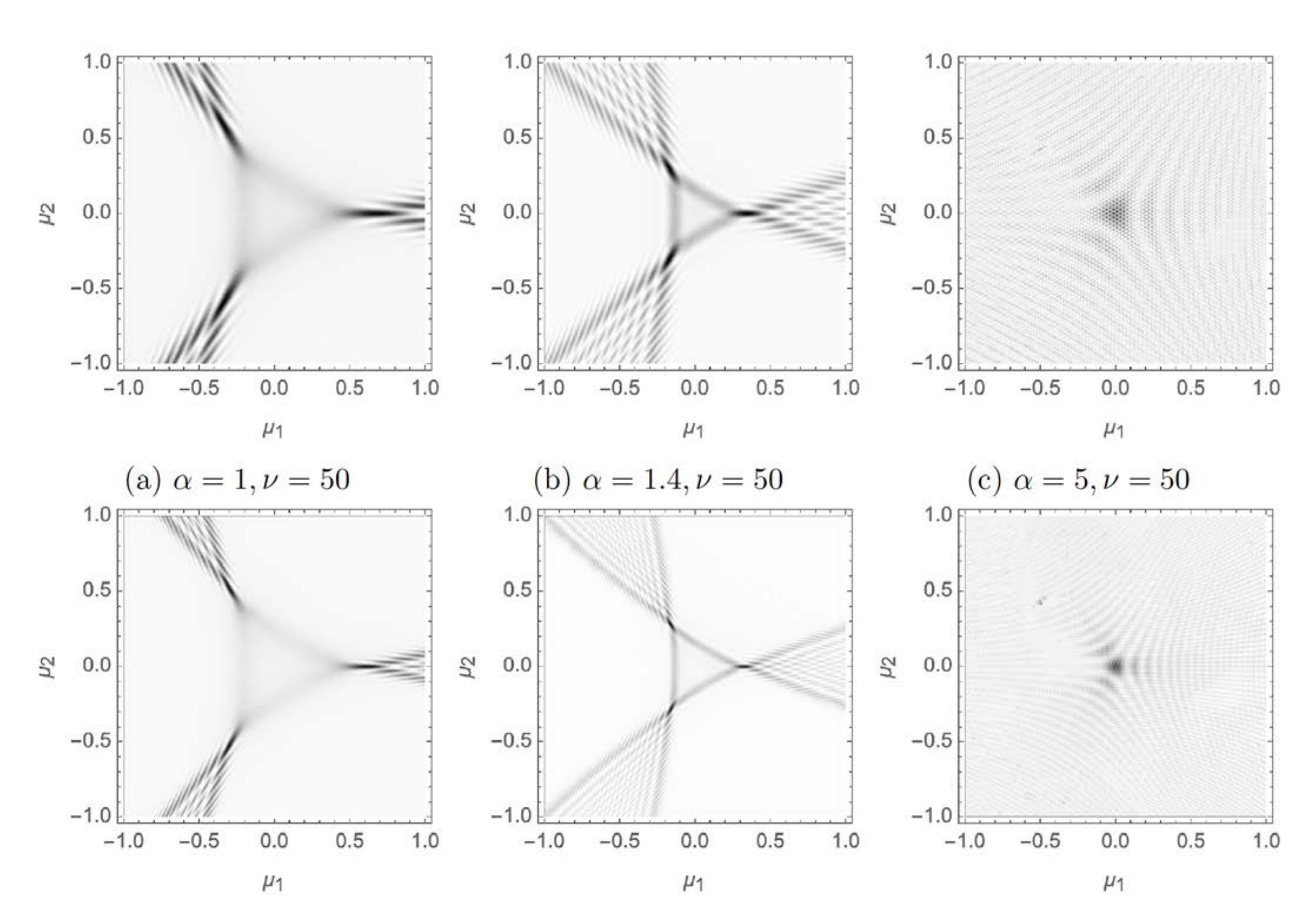


$$2d:$$

$$\Phi(\vec{x}) = (\vec{x} - \vec{\mu})^2 + \frac{\alpha(x_1^3 - 3x_1x_2^2)}{(1+x_1^2+x_2^2)}$$

Formation of elliptic umbilic catastrophe (when inner triangle shrinks to zero) via merger of three cusps

localised elliptic umbilic



(a) $\alpha = 1, \nu = 50$ (b) $\alpha = 1.4, \nu = 50$ (c) $\alpha = 5, \nu = 50$

(d) $\alpha = 1, \nu = 100$ (e) $\alpha = 1.4, \nu = 100$ (f) $\alpha = 5, \nu = 100$

c.f. beautiful works of Berry (thy) and Nye (expt) on diffraction catastrophes

(see e.g. Berry, M V, & Upstill, C, 1980 Progress in Optics XVIII, 257-346, ‘Catastrophe optics’)

Our method is more basic and enables one to tackle more generic cases

Summary

New approach to the *Lorentzian* (complex) path integral

Simple illustration: interference patterns with caustics, *e.g.*, in plasma lensing of pulsars/FRBs – path integrals in the sky!